LIFTS AND VERTEX PAIRS IN SOLVABLE GROUPS

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ABSTRACT. Suppose G is a p-solvable group, where p is odd. We explore the connection between lifts of Brauer characters of G and certain local objects in G, called vertex pairs. We show that if χ is a lift, then the vertex pairs of χ form a single conjugacy class. We use this to prove a sufficient condition for a given pair to be a vertex pair of a lift and to study the behavior of lifts with respect to normal subgroups.

1. Introduction

Throughout this paper G is a finite group and p is a fixed prime. If G is p-solvable and $\varphi \in \mathrm{IBr}_p(G)$, the Fong-Swan theorem shows that there is an irreducible character $\chi \in \mathrm{Irr}(G)$ such that $\chi^o = \varphi$, where χ^o denotes the restriction of χ to the p-regular elements of G. In this case the character χ is called a *lift* of φ, and in general, if χ^o is irreducible, then χ is called *a lift*. It is certainly the case that φ could have many lifts, and one active area of research is to understand the set of lifts of φ (see [1], [6], [21]).

Suppose that M is a normal subgroup of G and χ is a lift of φ. One would like to know under what conditions are the constituents of χ_M lifts? By Clifford's theorem, if one constituent of χ_M is a lift, then they all are. Note that it is certainly not the case that the constituents of χ_M must be lifts (see the example at the end of [4]), though if p is odd and G/M is a p-group, Navarro has shown that the constituents of χ_M must be lifts [21]. In this note, we find another sufficient condition that will imply that the constituents of χ_M are lifts.

Our condition makes use of vertex pairs. To each ordinary irreducible character χ of a p-solvable group, one can associate a vertex pair (Q, δ) to χ, where Q is a p-subgroup of G and $\delta \in \mathrm{Irr}(Q)$ (see Section 4 for the precise definition of a vertex pair). These vertex pairs generalize the vertex subgroups developed by Green [7] and share many of their properties. In fact, if p is odd and G is p-solvable, and $\chi \in \mathrm{Irr}(G)$ has vertex pair (Q, δ) and is a lift of $\varphi \in \mathrm{IBr}_p(G)$, then it is known that Q is a vertex for the irreducible module corresponding to φ, and δ is linear [21]. These vertex pairs are "local" objects that yield information on the lifts of the Brauer characters.

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With this definition, we can state our condition:

Theorem 1. *Let G be p-solvable where p is odd, and suppose $M \lhd G$. Let $\chi \in \mathrm{Irr}(G)$ be a lift of $\varphi \in \mathrm{IBr}_p(G)$ with vertex pair (Q, δ), and write $P = Q \cap M$ and $\lambda = \delta_P$. If P is abelian and λ is invariant in $\mathbf{N}_M(P)$, then the constituents of χ_M are lifts.*

In order to prove this theorem, we need to understand the connection between lifts and their vertex pairs. In particular, we need a uniqueness result regarding vertex pairs that generalizes the main result of [3]. Given an irreducible character χ of a p-solvable group, there are many different ways to associate a vertex pair to χ, and the resulting vertex pairs need not be conjugate [2]. We show that if p is odd and χ is a lift of a Brauer character φ, then in fact all of the vertex pairs for χ are conjugate, and thus it makes sense to speak of "the" vertex pair of the lift χ.

Theorem 2. *Let p be an odd prime and G a p-solvable group. Suppose that $\chi \in \mathrm{Irr}(G)$ is a lift of $\varphi \in \mathrm{IBr}_p(G)$. Then all of the vertex pairs for χ are conjugate.*

This theorem is known to be false if $p = 2$ (see [2]). It is also known that the conclusion need not hold if χ is not a lift (see [3]).

As mentioned before, it is known that if p is odd and G is p-solvable, then the vertex subgroup Q of any lift χ of $\varphi \in \mathrm{IBr}_p(G)$ is also the vertex subgroup of the irreducible module corresponding to φ. It is not known, however, which characters of Q can be vertex characters of lifts of φ. Our last main result gives a sufficient condition for a character δ of Q to be the vertex character of a lift of φ.

Theorem 3. *Let p be an odd prime and G a p-solvable group. Suppose $\varphi \in \mathrm{IBr}_p(G)$ has vertex subgroup Q, and let $\delta \in \mathrm{Irr}(Q)$. If Q is abelian and δ is invariant in $\mathbf{N}_G(Q)$, then there is a unique lift of φ with vertex pair (Q, δ).*

It is not yet known whether the hypotheses that Q is abelian can be weakened.

2. Restriction to normal subgroups

In this section, we prove Theorem 1 (assuming Theorems 2 and 3). Before we can prove Theorem 1 we need an easy lemma. We omit the proof of this lemma, as the first part is Theorem 3.2 of [5] (and can also be found in [17]), and the proof of the second part consists of the exact same argument used to prove Theorem 1.1 of [4] (which used the version of Theorem 2 when $|G|$ is assumed to be odd - note that we now know we need only assume that G is p-solvable and p is odd).

Lemma 2.1. *Let p be a prime and let G be a p-solvable group, and let $\varphi \in \mathrm{IBr}_p(G)$ have vertex subgroup Q. Suppose $M \lhd G$. Then:*

(a) *There is some constituent θ of φ_M that has vertex $Q \cap M$.*
(b) *If p is odd, and χ is a lift of φ with vertex pair (Q, δ), then some constituent of χ_M has vertex pair $(Q \cap M, \delta_{Q\cap M})$.*

We will need to make use of a result of Navarro [19] regarding relative defect zero characters.

Definition 2.2. *Let p be a prime. If G is a group with $N \lhd G$, and $\mu \in \mathrm{Irr}(N)$ and $\chi \in \mathrm{Irr}(G \mid \mu)$, then we say that χ has relative defect zero (with respect to p) if*

$$(\chi(1)/\mu(1))_p = |G : N|_p.$$

If the prime p is clear from the context, we will simply refer to the relative defect zero characters over μ. We will denote the relative defect zero characters of G lying over μ by $\mathrm{rdz}(G \mid \mu)$. Note that $\mathrm{rdz}(G \mid 1_N)$ consists of precisely the defect zero characters of G/N.

The following, which is a restatement of Theorem 2.1 of [19], will be key to our arguments. Note there are no conditions on the group G or the prime p (other than of course $|G|$ is finite).

Theorem 2.3. *Let G be a finite group and p a prime. Let $D \lhd G$ be a p-subgroup and let $\mu \in \mathrm{Irr}(N)$ be G-invariant. Then there is a bijection $\chi \to \chi_\mu$ from the defect zero characters of G/D to $\mathrm{rdz}(G \mid \mu)$. If μ is linear, then $\chi^o = \chi_\mu^o$.*

We have not taken the time to generalize Theorem 2.3 from Brauer characters to π-partial characters. It is for this reason that the results in this section and Section 3 are stated in terms of Brauer characters. However, the results in this section and Section 3 are true for π-partial characters with $2 \in \pi$.

We will need to make use of character triple isomorphisms. The definition and key results for character triple isomorphisms can be found in Definition 11.23 of [9] and the results in Chapter 11 after that. This next lemma gives a connection between lifts and character triple isomorphisms. The argument for this next lemma appeared in the proof of Corollary B of [6].

Lemma 2.4. *Let G be a p-solvable group, let N be a normal p'-subgroup of G, and let $\theta \in \mathrm{Irr}(N)$ be G-invariant. Then (G, N, θ) is character triple isomorphic to (G^*, N^*, θ^*) where N^* is a central, p'-subgroup of G^*. Let $\chi \in \mathrm{Irr}(G \mid \theta)$ correspond to $\chi^* \in \mathrm{Irr}(G^* \mid \theta^*)$. Then χ is a lift if and only if χ^* is a lift. Furthermore, suppose ψ is a lift of $\varphi \in \mathrm{IBr}_p(G)$, and let $\varphi^* = (\psi^*)^o$. Then the number of lifts of φ is equal to the number of lifts of φ^*.*

Proof. By Theorem 5.2 of [13], there is a character triple (G^*, N^*, θ^*) which is isomorphic to (G, N, θ) and where N^* is a central, p'-subgroup. Take H to be a Hall p-complement of G. Let H^* correspond to H, and note that H^* is a Hall p-complement of G^*. By the Fong-Swan theorem, χ^o is not irreducible if and only if there there exist characters α, β such that $\chi^o = \alpha^o + \beta^o$. This

occurs if and only if $\chi_H = \alpha_H + \beta_H$. Using the character triple isomorphism, this is equivalent to $(\chi^*)_{H^*} = (\alpha^*)_{H^*} + (\beta^*)_{H^*}$ and to $(\chi^*)^o = (\alpha^*)^o + (\beta^*)^o$. We conclude that χ^o is irreducible if and only if $(\chi^*)^o$ is irreducible.

Suppose ψ is a lift of φ, then we define $\varphi^* = (\psi^*)^o$. Notice that $\chi \in \mathrm{Irr}(G)$ is a lift of φ if and only if $\chi_H = \varphi_H$. It follows that χ is a lift of φ if and only if χ^* is a lift of φ^*, and so, the number of lifts of φ equals the number of lifts of φ^*. □

We will also need to understand how vertex pairs behave with respect to the character triple isomorphism. (We will need the basic results about p-special, p'-special, and p-factorable characters, see [10] for example.)

Lemma 2.5. *Let G be a p-solvable group and let N be a normal p'-subgroup of G, and suppose $\theta \in \mathrm{Irr}(N)$ is G-invariant. Let (G^*, N^*, θ^*) be an isomorphic character triple where N^* is a central p'-subgroup of G^*. If $\chi \in \mathrm{Irr}(G|\theta)$ is a lift with vertex pair (Q, δ), then there exists a subgroup $Q^* \cong Q$ of G^* and a character $\delta^* \in \mathrm{Irr}(Q^*)$ such that (Q^*, δ^*) is a vertex pair for χ^*. Moreover, δ is invariant in $\mathbf{N}_G(Q)$ if and only if δ^* is invariant in $\mathbf{N}_{G^*}(Q^*)$.*

Proof. Suppose that $\chi \in \mathrm{Irr}(G)$ is a lift with vertex pair (Q, δ). Then there is a subgroup U containing QN and a factorable character $\alpha\beta$ of U (where α is p'-special and β is p'-special) that induces χ, and Q is a Sylow p-subgroup of U and $\beta_Q = \delta$. Notice that α lies over θ and N is in the kernel of β, so $\alpha\beta \in \mathrm{Irr}(U|\theta)$. Thus $(\alpha\beta)^* \in \mathrm{Irr}(U^*|\theta^*)$ is a factorable character that induces χ^*. Write $(\alpha\beta)^* = \alpha_1\beta_1$ (where α_1 is p'-special and β_1 is p-special), let Q^* be a Sylow p-subgroup of U^*, and write $\delta^* = (\beta_1)_{Q^*}$. Then $Q \cong Q^*$, and (Q^*, δ^*) is a vertex pair for χ^*.

To complete the proof, we show that with the above notation, δ is invariant in $\mathbf{N}_G(Q)$ if and only if δ^* is invariant in $\mathbf{N}_{G^*}(Q^*)$. Notice that δ has a unique p-special extension $\epsilon \in \mathrm{Irr}(QN)$, and that δ is invariant in $\mathbf{N}_G(Q)$ if and only if ϵ is invariant in $\mathbf{N}_G(QN)$. Let $\widehat{\theta}$ denote the unique p'-special extension of θ to QN, and note that $\widehat{\theta}\epsilon \in \mathrm{Irr}(QN|\theta)$. Moreover, $\alpha\beta \in \mathrm{Irr}(U)$ lies over $\widehat{\theta}\epsilon$. Now δ is invariant in $\mathbf{N}_G(Q)$ if and only if $\widehat{\theta}\epsilon$ is invariant in $\mathbf{N}_G(QN)$, which occurs if and only if $(\widehat{\theta}\epsilon)^*$ is invariant in $\mathbf{N}_{G^*}((QN)^*)$ (by the properties of a character triple isomorphism), which occurs if and only if the p-special factor ϵ_1 of $(\widehat{\theta}\epsilon)^*$ is invariant in $\mathbf{N}_{G^*}((QN)^*)$. Note that ϵ_1 necessarily restricts to a vertex character for χ^* obtained from $(U^*, (\alpha\beta)^*)$, so $(\epsilon_1)_{Q^*} = \delta^*$. Finally, note that ϵ_1 is invariant in $\mathbf{N}_{G^*}((QN)^*)$ if and only if δ^* is invariant in $\mathbf{N}_{G^*}(Q^*)$, and we have proven the lemma.

□

We now turn to Theorem 1, and in fact, we prove more. Note that in the statement of the following theorem, we do not assume that the vertex subgroup Q is abelian, only that a relevant subgroup of Q is abelian. Also, since p is odd, then the vertex character for the lift χ is necessarily linear.

Theorem 2.6. *Let G be p-solvable where p is odd, and suppose $M \lhd G$. Let $\chi \in \mathrm{Irr}(G)$ be a lift of $\varphi \in \mathrm{IBr}_p(G)$ with vertex pair (Q, δ), and write $P = Q \cap M$ and $\lambda = \delta_P$. If P is abelian and λ is invariant in $\mathbf{N}_M(P)$, then the constituents of χ_M are lifts. Moreover, if λ is invariant in $\mathbf{N}_G(P)$ and ψ is a constituent of χ_M, then $G_\psi = G_{\psi^o}$.*

Proof. To prove the first statement, it is enough to show that some constituent of χ_M is a lift. There is some constituent ψ of χ_M such that the Clifford correspondent ρ of χ in $\mathrm{Irr}(G_\psi \mid \psi)$ has vertex pair (Q, δ). Notice that by part (b) of Lemma 2.1, ψ has vertex pair (P, λ). If G_ψ is proper in G, then by induction, we see that ψ is a lift.

Thus, we may assume that ψ is invariant in G, and note that this implies, by part (b) of Lemma 2.1, that ψ has vertex (P, λ). Let $N = \mathbf{O}_{p'}(M)$ and note that $N \lhd G$. Let α be a constituent of χ_N, and thus also of ψ_N. Let $T = G_\alpha$ and assume that $T < G$. By replacing α by a conjugate if necessary, we may assume that the Clifford correspondent $\mu \in \mathrm{Irr}(T \mid \alpha)$ of χ has vertex pair (Q, δ). Note that a Frattini argument (since ψ is invariant in G) shows that $G = MG_\alpha$. By induction, we see that the unique constituent ν of μ_{N_α} is a lift, and necessarily lies over α. Since $\nu^o \in \mathrm{IBr}_p(N_\alpha \mid \alpha)$ induces irreducibly to N, then $\psi = \nu^N$ is a lift.

Therefore we may assume that α is invariant in G. We may now use Lemma 2.4 and Lemma 2.5 to replace the triple (G, N, α) with an isomorphic character triple without losing the information about the lifts or their vertex pairs. Thus, we may assume that N is a central p'-subgroup of G (and hence, also central in M). Also, note that by part (a) of Lemma 2.1, some constituent θ of φ_M has vertex subgroup P.

Let $K \supseteq N$ be such that $K/N = \mathbf{O}_p(M/N)$. Since N is a central p'-subgroup in M, then $K = N \times S$, where $S = \mathbf{O}_p(M)$. Since P is a vertex subgroup of θ, then $S \subseteq P$. Also, since P is abelian, then $P \subseteq \mathbf{C}_M(S)$, so $PN/N \subseteq \mathbf{C}_{M/N}(SN/N) \subseteq SN/N$, where the last containment is by the Hall-Higman lemma. Therefore, $P \subseteq S$, and thus $P = S \lhd M$ and by assumption, λ is invariant in M. By Theorem 2.3, since θ is a Brauer character of M with vertex P, there is a unique character in $\mathrm{rdz}(M \mid \lambda)$ that lifts θ. However, we know that ψ has vertex (P, λ), and thus $\psi \in \mathrm{rdz}(M \mid \lambda)$, and therefore ψ is a lift of θ.

To prove the second statement, notice that we have shown that ψ has vertex pair (P, λ), and thus by Theorem 3, ψ is the unique lift of ψ^o with vertex pair (P, λ). It is clear that $G_\psi \subseteq G_{\psi^o}$. To prove the reverse containment, we may without loss of generality assume that ψ^o is invariant in G, and prove that ψ is invariant in G. Note that by a Frattini argument, $G = M\mathbf{N}_G(P)$. Since we are assuming $\mathbf{N}_G(P)$ stabilizes λ, then $G = M\mathbf{N}_G(P, \lambda)$. Let $g \in G$, and write $g = mn$, where $m \in M$ and $n \in \mathbf{N}_G(P, \lambda)$. Then $\psi^g = \psi^n$. But ψ^n is a lift of $(\psi^o)^n = \psi^o$ and has vertex pair $(P, \lambda)^n = (P, \lambda)$, and thus, by the uniqueness in Theorem 3, we see that $\psi^n = \psi$, and therefore $\psi^g = \psi$ and ψ is invariant in G. $\square$

3. Theorem 3

In this section we prove Theorem 3 (using Theorem 2). We note that the proof of Theorem 3 bears many similarities with the proof of Theorem 2.6.

Before beginning the proof, we show that it can certainly be the case that there are irreducible characters of a vertex subgroup Q that are not vertex characters of lifts. Let $p = 7$ and let Q have order 7, and let T have order 3 and act nontrivially on Q, and let $G = Q \rtimes T$ be the semidirect product. Let $\varphi \in \mathrm{IBr}_p(G/Q)$ be nontrivial, and note that φ must be linear and φ has vertex subgroup Q. If $\delta \in \mathrm{Irr}(Q)$ is nontrivial, then any character of G lying over δ must have degree 3 and thus cannot be a lift of φ. Thus there are no lifts of φ with vertex pair (Q, δ). This example shows that we cannot remove the hypothesis that δ is invariant in $\mathbf{N}_G(Q)$ from Theorem 3.

We now prove Theorem 3 of the introduction.

Proof of Theorem 3. Let $N = \mathbf{O}_{p'}(G)$, and let $\alpha \in \mathrm{Irr}(N)$ be a constituent of φ_N. Take T to be the stabilizer of α, and suppose that $T < G$. Replacing α by a conjugate if necessary, we may assume that Q is a vertex subgroup for the Clifford correspondent $\eta \in \mathrm{IBr}_p(T \mid \alpha)$ of φ. Clearly, the hypotheses are inherited by η in $\mathrm{IBr}_p(T)$, and thus, by induction, there is a unique lift $\psi \in \mathrm{Irr}(T)$ of η with vertex pair (Q, δ). Note that $\psi^G \in \mathrm{Irr}(G)$, and $\varphi = \eta^G = (\psi^o)^G = (\psi^G)^o$, and thus ψ^G is a lift of φ. By Theorem 2, any vertex pair for ψ is a vertex pair for ψ^G, and thus (Q, δ) is a vertex pair of ψ^G.

We still need to show that ψ^G is the unique such lift of φ. Suppose χ_1 and χ_2 are lifts of φ with vertex pair (Q, δ). Again, choose $\alpha \in \mathrm{IBr}_p(N)$ so that the Clifford correspondent η of φ has vertex subgroup Q. Now the Clifford correspondents ψ_1 and ψ_2 of χ_1 and χ_2 have vertex pairs (Q, δ_1) and (Q, δ_2), respectively. In light of Theorem 2, δ_1 and δ_2 are conjugate to δ via elements in $\mathbf{N}_G(Q)$. By assumption, $\mathbf{N}_G(Q)$ stabilizes δ, and thus $\delta_1 = \delta_2 = \delta$. Therefore, ψ_1 and ψ_2 are lifts of η with vertex pair (Q, δ), and thus, by induction, $\psi_1 = \psi_2$. We conclude that $\chi_1 = \chi_2$.

Therefore, we may assume that α is invariant in G. Using Lemma 2.4 and Lemma 2.5, we may replace (G, N, α) using a character triple isomorphism without losing information about the lifts or their vertex pairs, and thus, we may assume that N is a central p'-subgroup of G. Let $K \supseteq N$ be such that $K/N = \mathbf{O}_p(G/N)$. Since N is a central p'-subgroup in G, then $K = N \times P$, where $P = \mathbf{O}_p(G)$. Since Q is a vertex subgroup of φ, then $P \subseteq Q$. Also, since Q is abelian, then $Q \subseteq \mathbf{C}_G(P)$, so $QN/N \subseteq \mathbf{C}_{G/N}(PN/N) \subseteq PN/N$, where the last containment is by the Hall-Higman lemma. Therefore, $Q \subseteq P$, and thus $Q = P \lhd G$ and therefore by assumption, δ is invariant in G.

Now, $\varphi \in \mathrm{IBr}_p(G)$ has a normal vertex subgroup Q, and we may view φ as a Brauer character of G/Q which has defect zero. Thus, the unique character $\chi \in \mathrm{Irr}(G/Q)$ that lifts φ has defect zero. Applying Theorem 2.3, there is a unique character $\chi_\delta \in \mathrm{rdz}(G \mid \delta)$ that lifts φ. Since any lift of φ

that has vertex (Q, δ) must lie above δ and have relative defect zero, we are done. $\square$

4. Generalized vertices

In this section we will prove Theorem 2. Rather than work with Brauer characters, in this section we work in the context of Isaacs' partial characters to prove a slightly more general result. Hence, we will have a set of primes π. To define the π-partial characters, one needs to assume that G is π-separable. As in the context of Brauer characters, we let G^o denote the set of π-elements in G. Given an ordinary character χ, we use χ^o to denote the restriction of χ to G^o. The π-partial characters of G are the functions defined on G^o that are restrictions of ordinary characters. The π-partial characters that cannot be written as the sum of two other partial characters are called irreducible. We use $\mathrm{I}_\pi(G)$ to denote the irreducible π-partial characters of G. For a full exposition on π-partial characters, we refer the reader to [10] and [13].

The irreducible π-partial characters of G have many properties in common with the irreducible Brauer characters of a p-solvable group. (In fact, if $\pi = p'$, then $\mathrm{I}_\pi(G) = \mathrm{IBr}_p(G)$, and the requirement that p is odd is equivalent to $2 \in \pi$.) For example, we can define induction of partial characters from subgroups in the same way one defines induction of Brauer characters. Given an irreducible π-partial character φ of G, we can define a vertex Q for φ to be a Hall π'-subgroup of a subgroup U that contains a π-partial character κ of π-degree that induces φ. Isaacs and Navarro proved in [16] that all of the vertices for φ are conjugate in G. (A different proof of this fact is in [15].) There also exists a Clifford correspondence for π-partial characters. If G is π-separable and $N \lhd G$ and $\theta \in \mathrm{I}_\pi(N)$, then induction is a bijection from the set $\mathrm{I}_\pi(G_\theta \mid \theta)$ to $\mathrm{I}_\pi(G \mid \theta)$ (see [12]).

We also need to consider π-special characters. Let G be a π-separable group. A character $\chi \in \mathrm{Irr}(G)$ is π-special if $\chi(1)$ is a π-number and for every subnormal group M of G, each irreducible constituent of χ_M has determinantal order that is a π-number. Many of the basic results of π-special characters can be found in Section 40 of [8] and Chapter VI of [18]. One result that is proved is that if α is π-special and β is π'-special, then $\alpha\beta$ is necessarily irreducible. Furthermore, if α' is π-special and β' is π-special so that $\alpha'\beta' = \alpha\beta$, then $\alpha' = \alpha$ and $\beta' = \beta$. We say that χ is π-factored (or factored, if the π is clear from context) if $\chi = \alpha\beta$, where α is π-special and β is π'-special. Another result is that if H is a Hall π-subgroup of G, then restriction defines an injection from the π-special characters of G into $\mathrm{Irr}(H)$.

Following the terminology introduced in [3], we say (Q, δ) is a *generalized π-vertex* for $\chi \in \mathrm{Irr}(G)$ if there exists a pair (U, ψ) (where $U \subseteq G$ and $\psi \in \mathrm{Irr}(U)$) so that $\psi^G = \chi$, Q is a Hall π-complement of U, $\psi = \alpha\beta$ where

α is π-special and β is π'-special, and $\beta_Q = \delta$. In this context, we say that (U, ψ) is a *generalized π-nucleus* for χ.

In [3], the first author proved that if $|G|$ is odd and $\chi \in \mathrm{Irr}(G)$ is such that $\chi^o \in \mathrm{I}_\pi(G)$, then the generalized π-vertices for χ are conjugate. We now show that the hypothesis that $|G|$ is odd can be replaced by the hypothesis that G is π-separable and $2 \in \pi$. Our argument will parallel the argument in [3].

The main result, which is the π-version of Theorem 2 is the following.

Theorem 4.1. *Let π be a set of primes with $2 \in \pi$, and let G be a π-separable group. If $\chi \in \mathrm{Irr}(G)$ is such that $\chi^o \in \mathrm{I}_\pi(G)$, then all of the generalized π-vertices for χ are conjugate.*

The key to our work is a recent result of Navarro. Replacing p by a set of primes π with $2 \in \pi$, the proof of Lemma 2.1 of [21] proves:

Lemma 4.2. *Let π be a set of primes with $2 \in \pi$, and let G be a π-separable group. Let $\chi \in \mathrm{Irr}(G)$ be π'-special. If $\chi(1) > 1$, then χ^o is not in $\mathrm{I}_\pi(G)$.*

For the remainder of this section, our work will parallel the work in [3]. The following should be compared with Lemma 2.3 of [3].

Lemma 4.3. *Let π be a set of primes with $2 \in \pi$, and let G be a π-separable group. Let $\chi \in \mathrm{Irr}(G)$ be such that $\chi^o \in \mathrm{I}_\pi(G)$. If $U \leq G$ and $\psi \in \mathrm{Irr}(U)$ is a π-factored character that induces χ, then the π'-special factor of ψ is linear. Moreover, if Q is a Hall π-complement of U, then Q is a vertex subgroup of χ^o.*

Proof. Note that since $\chi^o \in \mathrm{I}_\pi(G)$, and $\psi^G = \chi$, then $\psi^o \in \mathrm{I}_\pi(U)$. Since ψ is π-factored, we have $\psi = \alpha\beta$ where α is π-special and β is π'-special. It follows that $\beta^o \in \mathrm{I}_\pi(U)$. By Lemma 4.2, $\beta(1) = 1$. It follows that ψ has π-degree and $\psi^o \in \mathrm{I}_\pi(U)$. By Theorem B of [16], Q is a vertex subgroup of χ^o. □

The next lemma is similar to Lemma 3.1 of [3].

Lemma 4.4. *Let π be a set of primes with $2 \in \pi$, and let G be a π-separable group. Let $\chi \in \mathrm{Irr}(G)$ be such that $\chi = \alpha\beta$ where α is π-special and β is linear and π'-special. Suppose $\psi \in \mathrm{Irr}(U)$ is π-factored and induces χ. If δ is the π'-special factor of ψ, then $\beta_U = \delta$.*

Proof. Note that $\alpha = \alpha\beta\beta^{-1} = \chi\beta^{-1}$. It follows that $(\psi\beta^{-1}|_U)^G = \psi^G\beta^{-1} = \chi\beta^{-1} = \alpha$. Since α is π-special, we may use Theorem C of [11] to see that $\psi\beta^{-1}|_U$ is π-special. We can write $\psi = \gamma\delta$ where γ is π-special. Now, $\gamma^o = \psi^o = (\psi\beta^{-1}|_U)^o$, and so, $\gamma = \psi\beta^{-1}|_U = \gamma\delta\beta^{-1}|_U$. It follows that $\delta\beta^{-1}|_U = 1_U$, and hence, $\delta = \beta_U$. □

The next result should be compared with Lemma 3.2 of [3]. Let π be a set of primes with $2 \in \pi$ and suppose G is π-separable. We will need the basic properties of the set $\mathrm{B}_\pi(G) \subseteq \mathrm{Irr}(G)$ introduced in [10]. In particular,

we need to know that restriction to G^o gives a bijection from $\mathrm{B}_\pi(G)$ to $\mathrm{I}_\pi(G)$ and that the π-special characters of G are precisely the characters of π-degree in $\mathrm{B}_\pi(G)$. We will also use the magic field automorphism that was described in [14]. We write σ to denote the magic field automorphism. Let $\chi \in \mathrm{Irr}(G)$. In [14], it is proved that $\chi \in \mathrm{B}_\pi(G)$ if and only if $\chi^\sigma = \chi$ and $\chi^o \in \mathrm{I}_\pi(G)$.

Lemma 4.5. *Let π be a set of primes with $2 \in \pi$, and let G be a π-separable group with subgroup U. Suppose $\chi \in \mathrm{Irr}(G)$ satisfies $\chi^o \in \mathrm{I}_\pi(G)$. Assume $\psi \in \mathrm{Irr}(U)$ is π-factored so that $\chi = \psi^G$. Suppose $|G : U|$ is a π-number and the π'-special factor of ψ extends to G. Then χ is π-factored.*

We will use the notation β' to denote the restriction of an ordinary character β of G to the π'-elements of G.

Proof. Let $\psi = \alpha\beta$ where α is π-special and β is π'-special. Let $\varphi = \beta' \in \mathrm{I}_{\pi'}(U)$. Let $\eta \in \mathrm{Irr}(G)$ be an extension of β. Now, $(\eta')_U = (\eta_U)' = \beta' \in \mathrm{I}_{\pi'}(U)$. It follows that $\eta' \in \mathrm{I}_{\pi'}(G)$. Let $\delta \in \mathrm{B}_{\pi'}(G)$ so that $\delta' = \eta'$. Observe that $\delta(1) = \eta(1) = \beta(1)$ is a π'-number, and so δ is π'-special. Also, $(\delta_U)' = \beta'$ implies that $\delta_U \in \mathrm{Irr}(U)$. By Theorem A of [11], δ_U is π'-special. This implies that $\delta_U = \beta$.

We now have $\chi = \psi^G = (\alpha\beta)^G = \alpha^G\delta$. This implies that $\alpha^G \in \mathrm{Irr}(G)$. Notice that $(\alpha^G)^\sigma = (\alpha^\sigma)^G = \alpha^G$. Also, $\chi^o = (\alpha^G\delta)^o = (\alpha^G)^o\delta^o \in \mathrm{I}_\pi(G)$, and so, $(\alpha^G)^o \in \mathrm{I}_\pi(G)$. It follows that α^G is π-special. We conclude that χ is π-factored. □

The next result is similar to Corollary 3.3 of [3].

Lemma 4.6. *Let π be a set of primes with $2 \in \pi$, and let G be a π-separable group. Let $\chi \in \mathrm{Irr}(G)$ be π-factored and have π-degree. Let N be a normal subgroup of G and suppose $\theta \in \mathrm{Irr}(N)$ is a constituent of χ_N. Let T be the stabilizer of θ in G. If $\psi \in \mathrm{Irr}(T \mid \theta)$ is the Clifford correspondent for χ with respect to θ, then ψ is π-factored.*

Proof. Observe that θ is π-factored. We can write $\chi = \gamma\delta$ and $\theta = \alpha\beta$ where γ and α are π-special and δ and β are π'-special. Since χ has π-degree, $\delta(1) = 1$ and thus, $\delta_N = \beta$. It follows that T is the stabilizer of α in G. We take $\mu \in \mathrm{Irr}(T \mid \alpha)$ to be the Clifford correspondent for γ with respect to α. We have $\gamma(1) = |G : T|\mu(1)$, and thus, $\mu(1)$ is a π-number. Observe that $(\mu^o)^G = (\mu^G)^o = \gamma^o \in \mathrm{I}_\pi(G)$ and thus, $\mu^o \in \mathrm{I}_\pi(T)$. Since $\alpha^\sigma = \alpha$, we have $\mu^\sigma \in \mathrm{Irr}(T \mid \alpha)$. Since $(\mu^\sigma)^G = (\mu^G)^\sigma = \mu^G$, it follows that $\mu^\sigma = \mu$, and we conclude that μ is π-special. Because δ is linear and π'-special, δ_T is π'-special. We see that $(\mu\delta_T)^G = \mu^G\delta = \gamma\delta = \chi$. Also, $(\mu\delta_T)_N = \mu_N\delta_N$, and so, $\alpha\beta = \theta$ is a constituent of $(\mu\delta_T)_N$. We obtain $\mu\delta_T \in \mathrm{Irr}(T \mid \theta)$. Since $(\mu\delta_T)^G = \chi = \psi^G$, we can use the Clifford correspondence to see that $\psi = \mu\delta_T$. Therefore, ψ is π-factored. □

Since the proof of the next lemma is essentially the proof of Lemma 3.4 of [3] where Lemma 4.3 is used in place of Lemma 2.3 of [3], we do not include it here.

Lemma 4.7. *Let π be a set of primes with $2 \in \pi$, and let G be a π-separable group. Let $\chi \in \mathrm{Irr}(G)$ be a lift of $\varphi \in \mathrm{I}_\pi(G)$, and suppose N is normal in G such that the constituents of χ_N are π-factored. Suppose $\psi \in \mathrm{Irr}(U)$ is π-factored, and suppose $\psi^G = \chi$. Then $|NU : U|$ is a π-number.*

This next lemma is similar to Lemma 3.5 of [3].

Lemma 4.8. *Let π be a set of primes with $2 \in \pi$, and let G be a π-separable group. Let $\chi \in \mathrm{Irr}(G)$ be a lift of $\varphi \in \mathrm{I}_\pi(G)$. Suppose ψ is a π-factored character of some subgroup H of G that induces χ, and suppose there is a normal subgroup N of G such that the constituents of χ_N are π-factored and $G = NH$. Then χ is π-factored and the π'-special factor of χ restricts irreducibly to the π'-special factor of ψ.*

Proof. Notice that the second conclusion follows from the first conclusion by Lemma 4.4. We assume the first conclusion is not true, and we take G, N, and H to be a counterexample with $|G : H| + |N|$ minimal.

By Lemma 4.3, the π'-special factor of ψ is linear, so $\psi(1)$ is a π-number. Applying Lemma 4.7, we see that $|G : H| = |HN : N|$ is a π-number. Since $\chi(1) = |G : H|\psi(1)$, we see that χ has π-degree.

Choose K normal in G so that N/K is a chief factor for G. Notice that the irreducible constituents of χ_K are π-factored. If $G = HK$, then G, K, and H form a counterexample with $|G : H| + |K| < |G : H| + |N|$ violating the choice of minimal counterexample. Thus, we have $HK < G$.

Notice that $G = NH = N(HK)$. Notice that $\psi^{HK} \in \mathrm{Irr}(HK)$ will be a lift of a partial character in $\mathrm{I}_\pi(HK)$. Also, the irreducible constituents of $(\psi^{HK})_K$ are constituents of χ_K, and thus must be factored. If $H < HK$, then $|HK : H| + |K| < |G : H| + |N|$, and so HK, K, and H cannot form a counterexample. Thus, ψ^{HK} must be factored and induce χ. Also, $|G : HK| + |N| < |G : H| + |N|$, so G, HK, and N do not form a counterexample. We conclude that χ is π-factored, a contradiction. This implies that $H = HK$.

We have $K \leq H$. Let η be an irreducible constituent of ψ_K. Notice that η^N has an irreducible constituent θ which is a constituent of χ_N, so θ and η are both π-factored. Since χ has π-degree, θ has a linear π'-special factor. If ν is the π'-special factor of η, then ν extends to both the π'-special factor of θ and the π'-special factor of ψ. This implies that ν is invariant in both N and H. Since $G = NH$, we conclude that ν is G-invariant.

Note that $|N : K|$ divides the π-number $|G : H|$ and thus $|N : K|$ is a π-number. Let $\hat{\nu}$ be the unique π'-special extension of ν to N, and since ν is G-invariant so is $\hat{\nu}$. We can now apply Corollary 4.2 of [10] to see that restriction defines a bijection from $\mathrm{Irr}(G \mid \hat{\nu})$ to $\mathrm{Irr}(H \mid \hat{\nu}_{N\cap H})$. Observe that the π'-special factor of ψ will belong to $\mathrm{Irr}(H \mid \hat{\nu}_{N\cap H})$ since $\hat{\nu}_{N\cap H}$ is

the unique π'-special extension of ν to $N \cap H$. It follows that the π'-special factor of ψ extends to G, and applying Lemma 4.5 we conclude that χ is factored, as desired. □

We make use of the normal nucleus constructed by Navarro in [20]. We quickly summarize this construction. Fix a character $\chi \in \mathrm{Irr}(G)$. Navarro shows that there is a unique subgroup N that is maximal subject to being normal in G and the irreducible constituents of χ_N are π-factored. If $N = G$, then take (G, χ) to be the normal nucleus of χ. If $N < G$, let θ be an irreducible constituent of χ_N. Navarro shows that in this case θ is not G-invariant. We then let $\chi_\theta \in \mathrm{Irr}(G_\theta \mid \theta)$ be the Clifford correspondent for χ with respect to θ. We define the normal nucleus for χ to be the normal nucleus of χ_θ which can be computed inductively since $G_\theta < G$. Note that the process terminates when we have a factorable character, and thus the normal nucleus character of χ is factorable and induces to χ. (The definition of the normal nucleus is obviously motivated by Isaacs' construction of the subnormal nucleus in [10].) It can be easily seen that all of the normal nuclei for χ are conjugate.

The proof of Theorem 4.1 is essentially the proof of Theorem 4.1 of [3], and thus we do not include it here in full detail. However, we do provide a brief sketch of the proof. The goal is to show that if (U, ψ) is any generalized π-nucleus of χ, then the generalized π-vertex of χ defined by (U, ψ) is conjugate to a vertex for χ arising from a normal nucleus. Lemmas 4.3 and 4.4 allow us to assume that χ is not factorable. Let $N \lhd G$ be maximal so that the constituents of χ_N are factorable. By Lemma 4.7, we see that $|NU : U|$ is a π-number, and thus Lemma 4.8 allows us to replace the pair (U, ψ) with (NU, ψ^{NU}), and thus we may assume $N \subseteq U$. Letting θ be a constituent of ψ_N, we use Lemma 4.6 and Lemma 4.4 to replace (U, ψ) with the pair (U_θ, ξ), where ξ is the Clifford correspondent for ψ in $\mathrm{Irr}(U_\theta|\theta)$. We finish by applying the inductive hypothesis to the group G_θ and the Clifford correspondent for χ lying over θ, which by definition has a normal nucleus in common with χ.

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